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A twist condition and periodic solutions of Hamiltonian systems

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Abstract

In this paper, we investigate existence of nontrivial periodic solutions to the Hamiltonian system

$$-J\dot{z} = H'(t,z), \quad z \in \mathbb{R}^{2N}. \tag{HS}$$

Under a general twist condition for the Hamiltonian function in terms of the difference of the Conley–Zehnder index at the origin and at infinity we establish existence of nontrivial periodic solutions. Compared with the existing work in the literature, our results do not require the Hamiltonian function to have linearization at infinity. Our results allow interactions at infinity between the Hamiltonian and the linear spectra. The general twist condition raised here seems to resemble more the spirit of Poincaré's last geometric theorem.

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1. Introduction

This paper is concerned with a classical problem on the existence of periodic solutions of Hamiltonian systems when a twist condition near the origin and infinity holds for the Hamiltonian function. More precisely, consider the system

$$-J\dot{z} = H'(t,z), \quad t \in \mathbb{R},\ z \in \mathbb{R}^{2N}, \tag{HS}$$

where J is the standard symplectic matrix in $\mathbb{R}^{2N}$, the Hamiltonian function $H \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ is τ-periodic in t for some $\tau > 0$ and satisfies, among other technical conditions, the following

$$H'(t,z) = B_0(t)z + o\big(|z|\big) \quad \text{as } |z| \to 0 \text{ uniformly in } t \in [0,\tau], \tag{1.1}$$

$$H'(t,z) = B_\infty(t)z + o\big(|z|\big) \quad \text{as } |z| \to \infty \text{ uniformly in } t \in [0,\tau] \tag{1.2}$$

for τ-periodic continuous symmetric matrix functions $B_0(t)$ and $B_\infty(t)$. Here and below we use H', H'' to denote the first and the second derivatives of H with respect to $z \in \mathbb{R}^{2N}$. Assume B_0 and B_∞ are non-degenerate in the sense that the linear systems $-J\dot{z} = B_i(t)z$ with $i = 0, \infty$ do not have 1 as a Floquet multiplier. Then the classical results due to Amann and Zehnder, Conley and Zehnder, Long and Zehnder [4,5,13,26] give the existence of a nontrivial τ periodic solution to the system if B_∞ cannot be continuously deformed to B_0 in the set of non-degenerate continuous loops of τ periodic symmetric matrices. The twisting is referred to the fact that they stand in different equivalent loop classes.

A quantitative way to measure the twisting is given by the Conley–Zehnder index (or Maslov index), which was introduced in [19], and developed in [13,26] for the study of Hamiltonian systems in relation with Morse theory and critical point theory. Let $Sp(N,\mathbb{R})$ be the set of all $2N \times 2N$ symplectic matrices, and let

$$\Gamma = \big\{\gamma \in C\big([0,\tau], Sp(N,\mathbb{R})\big) \bigm| \gamma(0) = I,\ \gamma(\tau) \text{ has no eigenvalue } 1\big\}$$

where I denotes the identity matrix in $\mathbb{R}^{2N}$. By the results of Conley and Zehnder [13] and Long and Zehnder [26], there is a map $j : \Gamma \to \mathbb{Z}$ satisfying that γ_1 and γ_2 lie in the same component of Γ if and only if $j(\gamma_1) = j(\gamma_2)$. For non-degenerate $B(t)$, one defines the Conley–Zehnder index for B, $i(B) = k$ if $j(W) = k$, where W is the fundamental solution matrix to the linear system $-J\dot{y} = B(t)y$. When B is degenerate, Long [24] extended the concept, and the Conley–Zehnder index for a τ-periodic continuous symmetric matrix function $B(t)$ is a pair of integers, denoted by $(i(B), \nu(B))$, where $i(B)$ is defined as lower limit of the indices of non-degenerate ones, and $\nu(B)$ is the dimension of the kernel of $W(\tau) - I$. Thus $B(t)$ is non-degenerate if and only if $\nu(B) = 0$, that is, the linear system $-J\dot{z} = B(t)z$ has only trivial τ periodic solution 0. We refer to the books of Abbondandolo [2] and Long [25] for a more detailed account of the concept.

Condition (1.2) is referred in the literature as asymptotically linear for the nonlinearity (or asymptotically quadratic for the Hamiltonian). In the pioneer work [4] of Amann and Zehnder in 1980, under a twist condition in terms of the Conley–Zehnder indices of B_0 and B_∞, namely,

$$\big|i(B_0) - i(B_\infty)\big| \geqslant 1, \tag{1.3}$$

the existence of a nontrivial τ periodic solution was proved when B_0 and B_∞ were assumed to be constant matrices and non-degenerate. This result was established by Conley and Zehnder [13] in 1984 for the case where B_0 and B_∞ are non-degenerate continuous τ periodic symmetric matrix functions in $\mathbb{R}^{2N}$ for $N \geqslant 2$, and by Long and Zehnder [26] in 1990 for the case $N = 1$. It is well known that the twist condition of the form (1.3) is related to the famous Poincaré–Birkhoff theorem on the existence of fixed points of area preserving homeomorphisms on an annulus under twist conditions on the opposite sides of the boundary. The twist condition for Hamiltonian systems is measured by the gap between the Conley–Zehnder index at zero and at infinity.

Generalizations of this type results to allow B_0 and B_∞ to be degenerate were given in [1,5,11, 12,18,21,22,28,34–36] and references therein. There are two types of results basically. Among these works we refer to [12] and references therein for one type of results that allow resonance at infinity under some additional technical conditions such as Landsmann–Lazer condition, Rabinowitz resonant condition, the strong resonant conditions, etc. The twist condition is assumed in the form $i(B_0) \notin [i(B_\infty), i(B_\infty) + \nu(B_\infty)]$ in case B_0 is non-degenerate. These technical assumptions at infinity are crucial to establish compactness for the problem. There have been a variety of variations on the conditions at infinity, such as those mentioned above (see e.g., [11,12,18,21,22,29,32–36] and references therein), some type of asymptotic conditions at infinity are needed in all these results, as far as we know. On the other hand, following a result of Benci and Fortunato [6] for second order systems, another type of results was given in [1] and [28] for the first order systems, which allow resonant linearization at infinity without the technical conditions as those mentioned above but require a stronger version of the twist condition, namely $i(B_0) \notin [i(B_\infty) - 1, i(B_\infty) + \nu(B_\infty) + 1]$.

In this paper we consider a more general form of twist condition between the origin and infinity for the Hamiltonian function and our condition does not require the Hamiltonian to have a linearization at infinity. Instead of (1.2), we assume that there exists a continuous τ periodic symmetric matrix function $B_\infty(t)$ such that for some $K > 0$

$$H''(t,z) \geqslant B_\infty(t) \quad \big(\text{resp., } H''(t,z) \leqslant B_\infty(t)\big), \quad \forall t \in [0,\tau],\ |z| \geqslant K, \tag{1.4}$$

where for two symmetric matrices A and B, $A \leqslant B$ means that $B - A$ is semi-positively definite. Our main result is that if $H'(t,z)$ is of linear growth in z, and (1.1) and (1.4) hold with B_0 and B_∞ being non-degenerate, the system (HS) has a nontrivial τ-periodic solution provided the following twist condition holds

$$i(B_\infty) > i(B_0) + 1 \quad \big(\text{resp., } i(B_\infty) < i(B_0) - 1\big). \tag{1.5}$$

This form of twist condition, with the Hessian at zero and infinity twisting toward opposite directions, seems to be more natural in the spirit of the original formulation of Poincaré's last geometric theorem.

The novelty of our result is that we do not need the system to be asymptotically linear at infinity like (1.2) which has been assumed in all works in the literature. Note that a stronger version of (1.2) would be $H''(t,z) = B_\infty(t) + o(1)$ as $|z| \to \infty$. Our condition (1.5) requires only a one-sided comparison between $H''(t,z)$ and $B_\infty(t)$ at infinity. The twisting between the origin and infinity is reflected in (1.5). Thus our results extend the aforementioned second type of results, in particular Theorem B in [1], to a wider class of Hamiltonian functions not necessarily having linearization at infinity. On the other hand, without requiring the usual technical conditions at infinity like in [11,12,18,22] our results under the stronger twist condition allow interactions between the nonlinearity and linear spectrum at infinity. Finally our approach can also be adopted for obtaining multiplicity results when a certain symmetry is present in the system.

The basic idea of our proof is to modify the original problem so that the modified problems are non-resonant at infinity and to get control of the solutions for the modified problems so that they are solutions of the original problem. In spirit the idea was already used in [1,6,28] in the setting of asymptotically linear problems, though technically our constructions of modifications are different from those in [1,6,28]. Our approach follows closely to that in [23] for nonlinear elliptic problems. Due to the strongly indefinite nature of Hamiltonian systems, we need to overcome some more involved issues. Let us outline our strategy here. We first modify the system (HS) by modifying the Hamiltonian function to get a family of approximate systems which are non-degenerate at infinity. Due to the indefinite nature of the problem, we make use of the saddle-point reduction process as developed by Amann and Zehnder [4], Chang [10], and Long [25]; here we need to carefully make a suitable choice of the reduction setting. For the approximate systems we make use of Morse theory [10,27] or minimax principle [20,29] to obtain solutions to the modified systems whose Morse indices can be estimated from the constructions. Finally in a crucial step we establish the L^∞ bound of the solutions constructed for the approximate systems and therefore obtain solutions to the original system.

This paper is organized as follows. In Section 2, we introduce some preliminaries including the saddle point reduction, and establish some further estimates that we need in the proofs. In Section 3, we first give the precise statement of our main results, then we present the proofs of the results. We conclude the paper with a few remarks in Section 4.

2. Preliminaries

In this section we recall the saddle point reduction procedure for the variational formulation of the system

$$-J\dot{z} = H'(t,z). \tag{HS}$$

The saddle point reduction method was first used by Amann [3], and then by Amann and Zehnder in their celebrated paper [4] to treat Hamiltonian systems. This was further simplified by Chang in [10] and recollected by Long in [25]. We first recall the basic materials from the books by Chang [10] and Long [25]. Then we further develop some estimates which we need in the proof of the main theorems.

In this section we always assume

(H_1) $H \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$, and for some $\tau > 0$, it is τ-periodic in t.
Without loss of generality, throughout the paper we assume $\tau = 2\pi$.

(H_2) There exists a constant $\hat{c} > 0$ such that

$$\left|H''(t,z)\right| \leqslant \hat{c}, \quad \forall (t,z) \in \mathbb{R} \times \mathbb{R}^{2N}.$$

It is well known that under (H_1) and (H_2), weak solutions of (HS) are classical. Thus we just need to consider the weak solutions. Let $S_{2\pi} = \mathbb{R}/(2\pi\mathbb{Z})$, $L = L^2(S_{2\pi}, \mathbb{R}^{2N})$ and let $E = W^{1,2}(S_{2\pi}, \mathbb{R}^{2N})$ be equipped with the usual norm. In space L we define an operator

$$Az = -J\dot{z},$$

then the domain of A is E. The spectrum of the operator A is $\sigma(A) = \mathbb{Z}$ with each eigenvalue being of multiplicity $2N$. The eigenspace of A corresponding to the eigenvalue $k \in \mathbb{Z}$ is

$$E_k = \exp(ktJ)\mathbb{R}^{2N} = \big((\cos kt)I + (\sin kt)J\big)\mathbb{R}^{2N}.$$

In particular

$$\ker A = E_0 = \mathbb{R}^{2N}.$$

Let

$$g(z) = \int_0^{2\pi} H\big(t, z(t)\big)\,dt, \quad z \in L.$$

Then $g \in C^1(L, \mathbb{R})$ and

$$g'(z) = H'(t, z),$$

$g'(z)$ is Gâteaux differentiable and its G-derivative is

$$dg'(z)y = H''(t, z)y.$$

Then (H_2) implies

$$\big\|dg'(z)\big\|_{\mathcal{L}(L)} \leqslant \hat{c}. \tag{2.1}$$

Define

$$f(z) = \frac{1}{2}\langle Az, z\rangle_L - g(z), \quad z \in E.$$

Under (H_1) and (H_2), $f \in C^1(E, \mathbb{R})$, and the following equation

$$Az = g'(z), \quad z \in E, \tag{2.2}$$

is the Euler equation of the functional f on L, i.e.

$$\big\langle f'(z), y\big\rangle_L = \big\langle Az - g'(z), y\big\rangle_L, \quad \forall y, z \in E.$$

Now in order to solve (2.2), we describe the saddle point reduction procedure as in [4,5,10, 25]. We follow [25] more closely.

Let

$$P_0 : L \to E_0 = \mathbb{R}^{2N}$$

be the projection map. Define the operator

$$A_0 z = Az + P_0 z, \quad z \in E.$$

Then A_0 is invertible.

Let $\hat{c}$ be as in (H_2) and choose

$$\beta > 2(\hat{c}+1), \quad \beta \notin \mathbb{Z}.$$

We define the following projections on the space L according to the spectral resolution of A_0

$$P = \int_{-\beta}^{\beta} dE_\lambda, \qquad P^+ = \int_{\beta}^{\infty} dE_\lambda, \qquad P^- = \int_{-\infty}^{-\beta} dE_\lambda,$$

$$Q^+ = \int_{0}^{\infty} dE_\lambda, \qquad Q^- = \int_{-\infty}^{0} dE_\lambda.$$

Then

$$L = L^+ \oplus L^- \oplus Z, \quad \text{where } Z = PL,\ L^\pm = P^\pm L.$$

Next let

$$S^+ = \int_{\beta}^{\infty} \lambda^{-\frac{1}{2}}\, dE_\lambda, \qquad S^- = \int_{-\infty}^{-\beta} (-\lambda)^{-\frac{1}{2}}\, dE_\lambda, \qquad R = \int_{-\beta}^{\beta} |\lambda|^{-\frac{1}{2}}\, dE_\lambda.$$

Then $S^\pm|_{L^\pm}$ and $R|_Z$ are injections. Let

$$V^\pm = S^\pm L^\pm, \qquad V^0 = RZ.$$

Define a subspace of L by

$$V = V^+ \oplus V^- \oplus V^0.$$

Then V is isometric to the Sobolev space $W^{\frac{1}{2},2}(S_{2\pi}, \mathbb{R}^{2N})$ under the following norm

$$\|v\|_V = \left(\left\|\left(S^+\right)^{-1} v^+\right\|_L^2 + \left\|\left(S^-\right)^{-1} v^-\right\|_L^2 + \left\|R^{-1} v^0\right\|_L^2 + \|v\|_L^2\right)^{\frac{1}{2}}$$

where

$$v = v^+ + v^- + v^0 \in V^+ \oplus V^- \oplus V^0 = V.$$

For $z = z^+ + z^- + x \in L^+ \oplus L^- \oplus Z = L$, we define

$$g_0(z) = g(z) + \frac{1}{2}\langle P_0 z, z\rangle_L$$

and

$$f_0(z) = \frac{1}{2}\big(\|z^+\|_L^2 + \|Q^+x\|_L^2 - \|Q^-x\|_L^2 - \|z^-\|_L^2\big) - g_0(v),$$

where $v = S^+z^+ + S^-z^- + Rx$. By the properties of the operators $S^\pm$, we have

$$f_0(z) = \frac{1}{2}\langle A_0 v, v\rangle_L - g_0(v) = \frac{1}{2}\langle Av, v\rangle_L - g(v) = f(v).$$

The Euler equation of the functional f_0 is given by

$$\begin{aligned} z^\pm &= \pm S^\pm g_0'(v), \\ Q^\pm x &= \pm Q^\pm R g_0'(v), \end{aligned} \tag{2.3}$$

where

$$v = S^+z^+ + S^-z^- + Rx.$$

By a contraction mapping argument, for each $x \in Z$ fixed, Eq. (2.3) can be solved uniquely, we denote the solution by

$$z^\pm = \xi^\pm(x).$$

It can be shown that

$$z^\pm \in C^1(Z, V).$$

Let

$$z(x) = w(x) + x$$

where

$$w(x) = w^+(x) + w^-(x) = S^+\xi^+\big(R^{-1}x\big) + S^-\xi^-\big(R^{-1}x\big),$$

then

$$z \in C^1(Z, E).$$

Now define the functional a on Z as

$$\begin{aligned} a(x) &= f_0\big(\xi^+\big(R^{-1}x\big) + \xi^-\big(R^{-1}x\big) + R^{-1}x\big) \\ &= \frac{1}{2}\big(\big\|\xi^+\big(R^{-1}x\big)\big\|_L^2 + \big\|Q^+\big(R^{-1}x\big)\big\|_L^2 - \big\|Q^-\big(R^{-1}x\big)\big\|_L^2 - \big\|\xi^-\big(R^{-1}x\big)\big\|_L^2\big) - g_0\big(z(x)\big). \end{aligned}$$

Then

$$a(x) = f\big(z(x)\big) \quad \text{for } x \in Z.$$

The above discussions can be summarized in the following theorem duo to Amann and Zehnder [4] and Chang [10] (see also the book by Long [25]).

Theorem 2.1. *Assume that H satisfies (H_1) and (H_2). Then there exist $a \in C^2(Z, \mathbb{R})$ and an injection map $z \in C^1(Z, L)$ such that $z : Z \to E = \operatorname{dom} A$ satisfies the following properties:*

(1) *The map z has the form $z = w(x) + x$, where $Pw(x) = 0$.*
(2) *The functional a satisfies*

$$a(x) = f\big(z(x)\big) = \frac{1}{2}\big\langle Az(x), z(x)\big\rangle_L - g\big(z(x)\big),$$
$$a'(x) = Ax - Pg'\big(z(x)\big) = Az(x) - g'\big(z(x)\big),$$
$$a''(x) = AP - Pdg'\big(z(x)\big)z'(x) = \big[A - dg'\big(z(x)\big)\big]z'(x)$$

and a' is globally Lipschitz continuous.

(3) *$x \in Z$ is a critical point of a if and only if $z(x)$ is a solution of $Az = g'(z)$.*

We need a further estimate on the map $z = z(x)$. We have

Lemma 2.2. *Assume that H satisfies (H_1) and (H_2) and $H'(t, 0) = 0$. Then in the setting above*

$$\big\|\xi^{\pm}(x)\big\| \leqslant \frac{2\sqrt{\beta}(\hat{c}+1)\tilde{c}}{\beta - 2(\hat{c}+1)}\|x\|, \quad \forall x \in Z,$$

where $\hat{c}$ is given in (H_2) and $\tilde{c} = \|R\|$ which is independent of β. Also we have

$$\big\|\big(\xi^{\pm}\big)'(x)\big\| \to 0, \quad \text{as } \beta \to \infty.$$

Proof. Note that

$$\xi^{\pm}(x) = \pm S^{\pm} g_0'(v), \qquad v = S^{+}\xi^{+}(x) + S^{-}\xi^{-}(x) + Rx.$$

Note that $H'(t, 0) = 0$ implies $g'(0) = 0$. Since

$$\big\|S^{\pm}\big\| \leqslant \frac{1}{\sqrt{\beta}},$$

we have by (2.1) that

$$\begin{aligned}
\big\|\xi^{\pm}(x)\big\| &\leqslant \big\|S^{\pm} g_0'(v)\big\| \\
&\leqslant \frac{1}{\sqrt{\beta}}\big\|g'(v) + P_0 v\big\| \\
&\leqslant \frac{\hat{c}+1}{\sqrt{\beta}}\|v\| \\
&\leqslant \frac{\hat{c}+1}{\sqrt{\beta}}\left(\frac{\|\xi^{+}(x)\|}{\sqrt{\beta}} + \frac{\|\xi^{-}(x)\|}{\sqrt{\beta}} + \tilde{c}\|x\|\right).
\end{aligned}$$

Therefore,

$$\left\|\xi^{+}(x)\right\|+\left\|\xi^{-}(x)\right\| \leqslant \frac{2\sqrt{\beta}(\hat{c}+1)\tilde{c}}{\beta-2(\hat{c}+1)}\|x\|.$$

Next, since

$$\left(\xi^{\pm}\right)'(x)=\pm S^{\pm} d g_0'(v)\left(S^{+}\left(\xi^{+}\right)'(x)+S^{-}\left(\xi^{-}\right)'(x)+R\right),$$

we have

$$\left\|\left(\xi^{+}\right)'(x)\right\|+\left\|\left(\xi^{-}\right)'(x)\right\| \leqslant \frac{2\sqrt{\beta}(\hat{c}+1)\tilde{c}}{\beta-2(\hat{c}+1)}, \quad \forall x \in Z. \qquad \square$$

Remark 2.3. For $w(x)$ we also have that there is a constant $C>0$ dependent of $\hat{c}$ but independent of β such that

$$\left\|w(x)\right\| \leqslant \frac{C}{\sqrt{\beta}}\|x\|, \qquad \left\|w'(x)\right\| \leqslant \frac{C}{\sqrt{\beta}}, \quad \forall x \in Z.$$

For each 2π-periodic continuous symmetric matrix function $B(t)$, one can associate a pair of integers $(i(B), \nu(B))$ to the linear system

$$-J\dot{y}=B(t)y, \quad y \in \mathbb{R}^N. \tag{2.4}$$

This pair of integers $(i(B), \nu(B))$ is called the Conley–Zehnder index of $B(t)$. Note that $\nu(B)=0$ if and only if (2.4) has only the trivial 2π periodic solution.

Now assume $z^*=z^*(t)$ is a solution of (HS). Then by the discussions above, $z^*=z^*(x^*)$ for some $x^* \in Z$, and x^* is a critical point of $a(x)$ in Z. Let

$$2d=\dim Z.$$

Denote by $m^-(x^*)$ and $m^0(x^*)$ the Morse index and nullity of x^* as a critical point of $a(x)$ in Z. Let $(i(z^*), \nu(z^*))$ be the Conley–Zehnder index for

$$B(t):=H''\left(t, z^*(t)\right).$$

We also call $(i(z^*), \nu(z^*))$ the Conley–Zehnder index of z^*. We shall need the following result from [25].

Theorem 2.4. *Under the assumptions* (H_1) *and* (H_2), *one has*

$$m^-\left(x^*\right)=i\left(z^*\right)+d, \qquad m^0\left(x^*\right)=\nu\left(z^*\right).$$

3. Main results and proofs

In this section we state the main results in this paper and then give the proofs. We further make the following assumptions.

(H_2^*) There exists $C_1 > 0$ such that for all (t, z), $|H'(t, z)| \leqslant C_1(1 + |z|)$.

(H_3) $H'(t, 0) = 0$ and denote the Conley–Zehnder index $(i(0), \nu(0))$ of $z = 0$ by (i_0, ν_0) and assume $\nu_0 = 0$.

($H_4^{\pm}$) There exists a 2π periodic symmetric matrix function $B_\infty(t)$ such that for some $K > 0$,

$$\pm H''(t, z) \geqslant \pm B_\infty(t), \quad \text{for all } t \in S_{2\pi},\ |z| \geqslant K.$$

Denote by (i_∞, ν_∞) the Conley–Zehnder index of B_∞ and assume $\nu_\infty = 0$.

Our results read as follows.

Theorem 3.1. *Assume (H_1), (H_2^*), (H_3) and (H_4^+) (resp. (H_4^-)). If $i_\infty > i_0 + 1$ (resp. $i_\infty < i_0 - 1$), then* (HS) *has at least one nontrivial 2π periodic solution.*

Theorem 3.2. *Assume (H_1), (H_2^*), (H_3) and (H_4^+) (resp. (H_4^-)). Assume in addition that H is even in z. If $i_\infty > i_0 + 1$ (resp. $i_\infty < i_0 - 1$), then* (HS) *has at least $|i_\infty - i_0| - 1$ pairs of nontrivial 2π periodic solutions.*

We remark that the same results hold for the resonant case $\nu_\infty \neq 0$ if we assume the condition $i_\infty > i_0 + 1$ (resp., $i_\infty + \nu_\infty < i_0 - 1$). This is consistent with the condition used in [1] for asymptotically linear systems. Thus Theorem 3.1 implies Theorem B in [1]. In our setting in $(H_4^{\pm})$ by considering $B_\infty(t) \mp \epsilon I$ instead of $B_\infty(t)$ for $\epsilon > 0$ small the resonant case at infinity can be reduced to non-resonant case. Thus we only give the proof for the non-resonant case here.

The outline of our proof is as follows.

(i) Consider a sequence of modified problems

$$-J\dot{z} = H_k'(t, z). \tag{HS$_k$}$$

Perform the saddle point reduction as done in Section 2 for (HS)$_k$ so that solutions of (HS)$_k$ correspond to critical points of the functional $a_{k,\beta_k} \in C^2(Z_{\beta_k}, \mathbb{R})$. This is done by choosing the parameter β_k suitably so that the estimates in steps (ii) and (iii) can be carried out.

(ii) For each $k \in \mathbb{N}$, use the finitely dimensional functional a_{k,β_k} to construct a nontrivial solution z_k whose Morse index can be controlled.

(iii) Establish L^∞ estimate for z_k so that for k large, z_k is a nontrivial solution of the original problem.

The cases of (H_4^+) and (H_4^-) are similar. In fact the case (H_4^-) follows from the case (H_4^+) applied to the Hamiltonian $-H(-t, z)$. We assume (H_4^+) from now on. We first construct the approximate Hamiltonian function sequences H_k.

Lemma 3.3. *Assume (H_1), (H_2^*) and (H_4^+). Then there exists a sequence of Hamiltonian functions*

$$H_k(t,z) \in C^2\big(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R}\big), \quad k\in\mathbb{N},$$

satisfying the following properties:

(a) *there exists an increasing sequence of real numbers $R_k \to \infty$ $(k\to\infty)$ such that*

$$H_k(t,z) \equiv H(t,z), \quad \forall |z| \leqslant R_k,\ t\in S_{2\pi};$$

(b) *for each $k = 1,2,3,\ldots,$*

$$H_k''(t,z) \geqslant B_\infty(t) \quad \forall |z| \geqslant K,\ t\in S_{2\pi};$$

(c) *there exist $C_1' > 0$ (independent of k) and $\hat{c}_k > 0$ such that for all $z\in\mathbb{R}^{2N}$, $t\in S_{2\pi}$, and $k\in\mathbb{N}$*

$$\big|H_k'(t,z)\big| \leqslant C_1'\big(1+|z|\big), \qquad \big|H_k''(t,z)\big| \leqslant \hat{c}_k;$$

(d) *there is $\gamma > 0$, $\gamma\notin\mathbb{Z}$, independent of k such that for each $k\in\mathbb{N}$ fixed,*

$$H_k(t,z) \sim \frac{\gamma}{2}|z|^2, \qquad H_k'(t,z)\sim \gamma z, \qquad H_k''(t,z)\sim\gamma I, \quad \text{as } |z|\to\infty.$$

Proof. The result is from our work [23]; here we just describe the constructions of such a sequence of Hamiltonian functions H_k for convenience.

Let K be as in (H_4). Choose a sequence $\{R_k\}$ of positive numbers such that

$$K < R_1 < R_2 < \cdots < R_k < \cdots \to \infty, \quad k\to\infty.$$

For each $k\in\mathbb{N}$, define $\phi_k : [R_k, 2R_k] \to \mathbb{R}$ as

$$\phi_k(s) = \frac{2}{9R_k^3}(s-R_k)^3 - \frac{1}{9R_k^4}(s-R_k)^4, \quad s\in[R_k,2R_k].$$

Then define the function

$$\psi_k(s) = 1 - \frac{128R_k^2}{9(12R_k^2+s^2)}.$$

Now for each $k\in\mathbb{N}$, define the function $\eta_k : [0,\infty)\to[0,1)$ by

$$\eta_k(s) = \begin{cases} 0, & 0\leqslant s\leqslant R_k,\\ \phi_k(s), & R_k\leqslant s\leqslant 2R_k,\\ \psi_k(s), & 2R_k\leqslant s<\infty.\end{cases}$$

Then the desired Hamiltonian functions H_k are defined by

$$H_k(t,z) = \bigl(1 - \eta_k\bigl(|z|\bigr)\bigr)H(t,z) + \frac{\eta_k(|z|)\gamma}{2}|z|^2, \quad k \in \mathbb{N},$$

which satisfy the properties (a)–(d). The verification of these properties is left to the interested readers. □

Now we give the proof of Theorem 3.1.

Proof of Theorem 3.1. We consider the case of (H_4^+) only. We break the proof into three parts as outlined before.

Part 1. For each $k \in \mathbb{N}$, we consider the modified problem

$$-J\dot{z} = H_k'(t,z), \tag{HS$_k$}$$

where H_k is given in Lemma 3.3. By the construction, we see that each H_k satisfies (H_1) and (H_2) with $\hat{c}$ being replaced with $\hat{c}_k$ from Lemma 3.3. Therefore we can perform on $(\mathrm{HS})_k$ the saddle point reduction procedure as described in Section 2. We will choose the number $\beta = \beta_k$ large according to k, which is used in the projection for the reductions. First we choose $\beta_k > 4(\hat{c}_k + 1)$ and $\beta_k \notin \mathbb{Z}$ as in Section 2. Thus for each k and such a β_k fixed, by Theorem 2.1, we have a functional

$$a_{k,\beta_k}(x), \quad x \in Z = Z_{\beta_k},$$

whose critical points give rise to solutions of $(\mathrm{HS})_k$. By Theorem 2.1, the functional $a_{k,\beta_k} \in C^2(Z_{\beta_k}, \mathbb{R})$. And there exists a map $z_{k,\beta_k} \in C^1(Z_{\beta_k}, L)$, $z_{k,\beta_k}(x) = w_{k,\beta_k}(x) + x$ with $w_{k,\beta_k}(x) \in E$, such that x is a critical point of a_{k,β_k} if and only if $z_{k,\beta_k}(x)$ is a solution of $(\mathrm{HS})_k$. Furthermore for some $C_k > 0$ only dependent of $\hat{c}_k$ but not dependent of β_k, it holds

$$\bigl\| w_{k,\beta_k}'(x) \bigr\| \leqslant \frac{C_k}{\sqrt{\beta_k}}, \quad \forall x \in Z_{\beta_k}.$$

Using the same decomposition of L, we can also perform the saddle point reduction procedure on the linear system

$$-J\dot{z} = B_\infty(t)z$$

which has only trivial solution $z = 0$ with the Conley–Zehnder index $(i_\infty, 0)$. Then we have a functional $a_{\infty,\beta_k} \in C^2(Z_{\beta_k}, \mathbb{R})$, an injection map $z_{\infty,\beta_k} \in C^1(Z_{\beta_k}, L)$, $z_{\infty,\beta_k}(x) = w_{\infty,\beta_k}(x) + x$ with $w_{\infty,\beta_k}(x) \in E$, and for some $C > 0$ independent of k, such that

$$\bigl\| w_{\infty,\beta_k}'(x) \bigr\| \leqslant \frac{C}{\sqrt{\beta_k}}, \quad \forall x \in Z_{\beta_k}.$$

Also we have for $\varphi \in Z_{\beta_k}$,

$$\bigl\langle a_{\infty,\beta_k}''(0)\varphi, \varphi \bigr\rangle = \bigl\langle \bigl(A - B_\infty(t)\bigr)\bigl(w_{\infty,\beta_k}'(0) + id\bigr)\varphi, \varphi \bigr\rangle.$$

Denote by $Z^-_{\beta_k}$ the negative eigenspace of $a''_{\infty,\beta_k}(0)$ in Z_{β_k}. Now we claim that there exist $r_0 > 0$, $\beta_0 > 0$ such that for all $\beta_k > \beta_0$ with $\beta_k \notin \mathbb{N}$,

$$\langle a''_{\infty,\beta_k}(0)\varphi, \varphi\rangle \leqslant -r_0\|\varphi\|^2, \quad \forall \varphi \in Z^-_{\beta_k}.$$

By contradiction, if the claim does not hold, we would have the sequences

$$\varphi_n \in Z^-_{\beta_n}, \quad \|\varphi_n\| = 1, \quad \beta_n \to \infty \text{ as } n \to \infty$$

such that

$$\big\langle \big(A - B_\infty(t)\big(w'_{\infty,\beta_n}(0) + id\big)\big)\varphi_n, \varphi_n\big\rangle \to 0 \quad \text{as } n \to \infty.$$

Without loss of generality, we may assume

$$A\varphi_n - P_n B_\infty(t)\big(w'_{\infty,\beta_n}(0) + id\big)\varphi_n = \alpha_n\varphi_n, \tag{3.1}$$

where

$$P_n = P_{\beta_n} := \int_{-\beta_n}^{\beta_n} dE_\lambda, \quad \alpha_n \to 0 \text{ as } n \to \infty.$$

Take $a \notin \mathbb{Z}$. Then the operator $A + a$ is invertible and $(A + a)^{-1}$ is compact. From (3.1) we see that

$$\varphi_n = (A + a)^{-1}\big(P_n B_\infty(t)\varphi_n + P_n B_\infty(t) w'_{\infty,\beta_n}(0)\varphi_n + (\alpha_n + a)\varphi_n\big). \tag{3.2}$$

Since $\{P_n B_\infty(t)\varphi_n + P_n B_\infty(t) w'_{\infty,\beta_n}(0)\varphi_n + (\alpha_n + a)\varphi_n\}$ is bounded, we see that $\{\varphi_n\}$ has a convergent subsequence which is still denoted by $\{\varphi_n\}$. Let

$$\varphi_n \to \varphi^* \quad \text{as } n \to \infty.$$

Then $\|\varphi^*\| = 1$. Passing to a limit in (3.2) and using

$$P_n B_\infty(t)\varphi_n \to B_\infty(t)\varphi^*, \qquad P_n B_\infty(t) w'_{\infty,\beta_n}(0)\varphi_n \to 0,$$

we obtain

$$A\varphi^* - B_\infty(t)\varphi^* = 0,$$

which is a contradiction since $\nu_\infty = 0$. Then we see that there exists $\beta'_0 \geqslant \beta_0$ such that for all $\beta_k > \beta'_0$ with $\beta_k \notin \mathbb{Z}$ and all $\varphi \in Z^-_{\beta_k}$,

$$\big\langle \big(A - B_\infty(t)\big)\varphi, \varphi\big\rangle \leqslant -\frac{r_0}{2}\|\varphi\|^2.$$

At this time we choose β_k large such that it holds $\|w'_{k,\beta_k}(x)\| \leqslant r_0/4\hat{c}_k$ for $x \in Z_{\beta_k}$. For this we may assume $\hat{c}_k\tilde{c} > r_0$ and we can choose $\beta_k > (\frac{16\hat{c}_k(\hat{c}_k+1)\tilde{c}}{r_0})^2$. From now on we choose β_k such that

$$\beta_k > \max\left\{4(\hat{c}_k+1), \beta'_0, \left(\frac{16\hat{c}_k(\hat{c}_k+1)\tilde{c}}{r_0}\right)^2\right\}, \quad \beta_k \notin \mathbb{Z}.$$

Part 2. Note that $x = 0$ is a critical point of $a_{k,\beta_k}(x)$. By Theorem 2.4, the Morse index of 0 for a_{k,β_k} is $i_0 + d_{\beta_k}$, where

$$2d_{\beta_k} = \dim Z_{\beta_k}.$$

By the result of Long [25], the Poincaré polynomial for the level sets of a_{k,β_k} is $t^{d_{\beta_k}+i_\gamma}$, here we use (i_γ, ν_γ) to denote the Conley–Zehnder index of 0 for the system

$$-J\dot{y} = H''_\gamma(0)y,$$

where

$$H_\gamma(z) = \gamma/2|z|^2$$

and γ is given in Lemma 3.3 so that $\nu_\gamma = 0$. By construction, we have that

$$i_\gamma \geqslant i_\infty.$$

Now by Morse inequality, a_{k,β_k} has a nontrivial critical point x_k with its Morse index satisfying

$$m^-(x_k) \leqslant d_{\beta_k} + i_0 + 1.$$

By Theorem 2.1(3),

$$z_k = z_{k,\beta_k}(x_k) = w^+_{k,\beta_k}(x_k) + w^-_{k,\beta_k}(x_k) + x_k$$

is a solution of the modified system $(\mathrm{HS})_k$.

Part 3. Finally we show that there exists $C > 0$ independent of k such that

$$\|z_k\|_{L^\infty} \leqslant C$$

and therefore z_k is a solution of the original problem (HS) for k large. To this end, we use an indirect argument. Assume

$$\|z_k\|_{L^\infty} \to \infty \quad \text{as } k \to \infty.$$

As z_k solves $(\mathrm{HS})_k$ and the gradients of the Hamiltonian functions H'_k have a uniform growth control, we see that

$$\|z_k\|_{L^\infty} \to \infty \quad \Longleftrightarrow \quad \|z_k\|_V \to \infty \quad \text{as } k \to \infty.$$

Now multiplying the system $(\mathrm{HS})_k$ by $z_k^+ - z_k^-$ where $z_k^\pm = Q^\pm z_k$, we get

$$\|z_k^+\|_V^2 + \|z_k^-\|_V^2 \leqslant C\|z_k\|_{L^2}^2.$$

Set

$$y_k = \frac{z_k}{\|z_k\|_V}.$$

By the compact embedding $W^{\frac{1}{2},2}(S_{2\pi}, \mathbb{R}^{2N}) \hookrightarrow L^s(S_{2\pi}, \mathbb{R}^{2N})$ for $s \geqslant 1$, we have for some $y \in V$,

$$y_k \rightharpoonup y \quad \text{in } V, \qquad y_k \to y \quad \text{in } L, \quad \text{and} \quad y \neq 0.$$

Note that y_k satisfies

$$-J\dot{y}_k = \frac{H_k'(t, z_k)}{\|z_k\|_V} =: G_k\big(t, y_k(t)\big).$$

By (H_2^*), (H_3) and Lemma 3.3 there is $C_1' > 0$ such that $|H_k'(t,z)| \leqslant C_1'|z|$. Thus we have for all k, $|G_k(t,y)| \leqslant C_1'|y|$. By regularity results (e.g., [8]), $\{y_k\}$ is bounded in C^α for some $\alpha > 0$. Therefore up to a subsequence we may assume y_k converges in uniform norm to y. We claim $y(t) \neq 0$ for all $t \in S_{2\pi}$. We show this by showing if $y(t_0) = 0$ for some t_0 then $y(t) = 0$ for all t. In fact, since

$$y_k(t) = y_k(t_0) + \int_{t_0}^{t} JG_k\big(s, y_k(s)\big)\,ds,$$

we have $\max_{t\in I_0} |y_k(t)| \leqslant 2|y_k(t_0)|$, where $I_0 = \{t \mid |t - t_0| \leqslant \frac{1}{2C_1'}\}$. Sending $k \to \infty$ we have $y(t) = 0$ for $t \in I_0$. Repeating this gives the claim.

Now for k large we have $\min_{t\in[0,2\pi]} |z_k(t)| \geqslant K$. Note that

$$\left|\int_0^{2\pi} H_k''\big(t, z_{k,\beta_k}(x_k)\big) w_{k,\beta_k}'(x_k)\varphi\varphi\,dt\right| \leqslant \frac{r_0}{4}\|\varphi\|^2, \quad \forall \varphi \in Z_{\beta_k}.$$

For any $\varphi \in Z_{\beta_k}$,

$$\begin{aligned}
\big\langle a_{k,\beta_k}''(x_k)\varphi, \varphi\big\rangle &= \big\langle \big(A - H_k''\big(t, z_{k,\beta_k}(x_k)\big)\big)\big(w_{k,\beta_k}'(x_k) + id\big)\varphi, \varphi\big\rangle \\
&= \big\langle \big(A - H_k''\big(t, z_{k,\beta_k}(x_k)\big)\big)\varphi, \varphi\big\rangle - \int_0^{2\pi} H_k''\big(t, z_{k,\beta_k}(x_k)\big) w_{k,\beta_k}'(x_k)\varphi\varphi\,dt.
\end{aligned}$$

Then when k is large enough for any $\varphi \in Z_{\beta_k}^-$

$$\begin{aligned}\langle a''_{k,\beta_k}(x_k)\varphi,\varphi\rangle &\leqslant \langle (A-B_\infty(t))\varphi,\varphi\rangle + \frac{r_0}{4}\|\varphi\|^2\\ &\leqslant -\frac{r_0}{4}\|\varphi\|^2.\end{aligned}$$

This gives that the Morse index $m^-(x_k)$ of x_k for the functional a_{k,β_k} satisfies

$$m^-(x_k) \geqslant \dim Z^-_{\beta_k}.$$

Since

$$\dim Z^-_{\beta_k} = d_{\beta_k} + i_\infty,$$

we see that

$$m^-(x_k) \geqslant d_{\beta_k} + i_\infty,$$

which contradicts to

$$m^-(x_k) \leqslant d_{\beta_k} + i_0 + 1 < d_{\beta_k} + i_\infty.$$

Therefore, solutions $z_k = z_{k,\beta_k}(x_k)$ are uniformly bounded in L^∞ and are solutions of (HS) for k large. The proof of Theorem 3.1 is finished. □

The proof of Theorem 3.2 is similar to the above. The difference is in the second part, instead of Morse theory we make use of minimax arguments for multiplicity of critical points in the presence of symmetry. We need to control Morse indices of critical points obtained for the even functional $a_{k,\beta}$. We state two results of this type from Ghoussoub [20] and Chang [10] (see also discussions about these results in [23]).

Let X be a Hilbert space and assume $\phi \in C^2(X,\mathbb{R})$ is an even functional, satisfies the (PS) condition, and $\phi(0)=0$. Let $\mathcal{K}$ be the set of critical points of ϕ. Denote $S_a = \{u \in X \mid \|u\| = a\}$.

Lemma 3.4. *(See [20, Corollary 10.19].) Assume Y and Z are subspaces of X satisfying* $\dim Y = j > k = \operatorname{codim} Z$. *If there exist $R > r > 0$ and $\alpha > 0$ such that*

$$\inf \phi(S_r \cap Z) \geqslant \alpha, \qquad \sup \phi(S_R \cap Y) \leqslant 0,$$

then ϕ has $j-k$ pairs of nontrivial critical points $\{\pm u_1, \pm u_2, \ldots, \pm u_{j-k}\}$ so that $\mu(u_i) \leqslant k+i$ for $i = 1,2,\ldots,j-k$.

Lemma 3.5. *(See [10, Corollary II 4.1].) Assume Y and Z are subspaces of X satisfying* $\dim Y = j > k = \operatorname{codim} Z$. *If there exist $r > 0$ and $\alpha > 0$ such that*

$$\inf \phi(Z) > -\infty, \qquad \sup \phi(S_r \cap Y) \leqslant -\alpha,$$

then ϕ has $j-k$ pairs of nontrivial critical points $\{\pm u_1, \pm u_2, \ldots, \pm u_{j-k}\}$ so that $\mu(u_i) + \nu(u_i) \geqslant k+i-1$ for $i = 1,2,\ldots,j-k$.

For the proof of Theorem 3.2 we follow the proof of Theorem 3.1 closely. Part 1 is the same. In Part 2 we use the above two lemmas to claim multiplicity of nontrivial critical points of $a_{k,\beta}$ (at least $|i_\infty - i_0|$ pairs) whose Morse indices are suitably controlled. In Part 3, we follow the same arguments to conclude that at least $|i_\infty - i_0| - 1$ pairs of these solutions are uniformly bounded in k and therefore they are solutions of the original systems for k large.

4. Further results and remarks

When the Hamiltonian function H is independent of t, i.e., for autonomous systems, the existence results may give constant solutions, i.e., equilibrium points of the system. There are standard conditions which assure that the solutions obtained are non-constant solutions. On the other hand, for autonomous systems there is a natural S^1 group action in the space and the variational formulations are invariant under the group action. This can be used to obtain multiplicity results (as in the case of the Hamiltonian being even in z in Section 3). Our methods can be combined with the existing techniques to extend these multiplicity results to the more general situations we consider in this paper. We state one such a result below.

Theorem 4.1. *Assume* (H_1), (H_2^*), (H_3) *and* (H_4^+) *(resp.* (H_4^-)*). Assume that* H *is independent of* t *and strictly convex in* z*. If* $i_\infty > i_0 + 1$ *(resp.* $i_\infty < i_0 - 1$*), then* (HS) *has at least* $\frac{1}{2}|i_\infty - i_0| - 1$ *non-constant geometrically different* 2π *periodic solutions.*

Here we say two solutions are geometrically different if one is not obtained by time re-scaling of the other.

Remark 4.2. We may impose additional conditions to guarantee the solutions obtained have minimal period 2π. The existence of periodic solutions with prescribed periods has been intensively studied in the last thirty years. Since the celebrated work of Rabinowitz [30] variational methods have been used to deal with existence of periodic solutions (see for instance [1,4–7,9,11–18,21, 22,24,26,34–36], monographs and survey papers [2,10,20,25,27,31–33] and references therein). There have been many works giving conditions to assure existence of solutions with a prescribed minimal period such as [14]. The conditions and techniques used to establish these results can be combined with our methods to obtain results for minimal periods in the situations we consider here.

Remark 4.3. As was discussed and demonstrated in [4,5,13] the twist condition (1.3) (together with (1.1), (1.2)) is related to the famous Poincaré–Birkhoff theorem which assures the existence of two fixed points for any area preserving homeomorphisms f on an annulus that satisfy a boundary twist condition. This boundary twist condition states that f advances points on the outer edge of the annulus positively and points on the inner edge negatively. Thus the map twists on the inner and outer edges in opposite direction. Results on existence of periodic solutions for Hamiltonian systems under twist conditions are regarded as higher dimensional generalizations of the Poincaré–Birkhoff theorem. In some sense the twist condition we proposed here, i.e., one-sided condition (1.5) (together with (1.1) and (1.4)), resembles more naturally this spirit. On the other hand, we need a stronger twisting in that $i_\infty > i_0 + 1$ (or $i_\infty < i_0 - 1$). It would be interesting to see whether this is necessary.

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